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ARITHMETIC PROPERTIES OF PARTITIONS WITH 5-CORES

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Abstract. Let $a_5(n)$ denote the number of partitions of n that are 5-cores. By using theta function identities, we find several infinite families of congruences modulo 3, 4 and 8 for $a_5(n)$.

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1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. A partition of n is said to be t-core if it has no hook numbers that are multiples of t. Let $a_t(n)$ denote the number of t-core partitions of n. The generating function of $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{f_t^t}{f_1}. \tag{1}$$

Here and throughout this paper, we use the notation

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and for any positive integer } k, \quad f_k := (q^k;q^k)_\infty.$$

Several mathematicians have obtained numerous congruence properties of t-core partition function (see, for example, [2, 6, 7, 14]). Ramanujan [13, p. 139] recorded the identity

$$q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=0}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}.$$

Garvan, Kim and Stanton [4] used the above identity to obtain the following theorem.

Theorem 1.1. *If the prime factorization of $n+1$ is*

$$n+1 = 5^\alpha \prod_{p_i\equiv\pm1 \pmod 5} p_i^{a_i} \prod_{q_j\equiv\pm2 \pmod 5} q_j^{b_j},$$

then

$$a_5(n) = 5^\alpha \prod_{p_i\equiv\pm1 \pmod 5} \frac{p_i^{a_i+1}-1}{p_i-1} \prod_{q_j\equiv\pm2 \pmod 5} \frac{q_j^{b_j+1}+(-1)^{b_j}}{q_j+1}.$$

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By using Theorem 1.1, one can easily obtain several arithmetic identities for $a_5(n)$. For instance, for all α, $n \geq 0$ and for any prime $p \equiv \pm 2 \pmod 5$,

$$a_5(p^\alpha n + p^\alpha - 1) = \frac{p^\alpha - (-1)^\alpha}{p+1} a_5(pn + p - 1) + \frac{p^\alpha + p(-1)^\alpha}{p+1} a_5(n)$$

and if $p \nmid (n+1)$, then

$$a_5(p^\alpha n + p^\alpha - 1) = \frac{p^{\alpha+1} + (-1)^\alpha}{p+1} a_5(n). \tag{2}$$

Radu and Sellers [11] obtained several congruences modulo small powers of 2 for $(2k+1)$-core partitions, by using the theory of modular forms. For example, they proved that

$$a_5(10n+2) \equiv a_5(10n+6) \equiv 0 \pmod 2, \tag{3}$$

for all nonnegative integers n. Chen [2] found many new congruences for p-core partition functions when $5 \leq p \leq 47$. For instance, if n is a quadratic nonresidue modulo 15, then

$$a_5(n-1) \equiv 0 \pmod 3.$$

With this motivation, we establish several new infinite families of congruences modulo 3, 4 and 8 for the partition function $a_5(n)$. For example, we have the following result.

Theorem 1.2. *For all integers $\alpha, \beta, \gamma, n \geq 0, u \in \{11, 13, 37, 59\}$ and $v \in \{73, 97\}$, we have*

$$a_5(2^{\alpha+2} \cdot 3^{\beta+1} \cdot 5^{\gamma+1} n + u \cdot 2^\alpha \cdot 3^\beta \cdot 5^\gamma - 1) \equiv 0 \pmod{12} \tag{4}$$

and

$$a_5(5 \cdot 2^{\alpha+3} \cdot 3^{2\beta+1} n + v \cdot 2^\alpha \cdot 3^{2\beta} - 1) \equiv 0 \pmod{24}. \tag{5}$$

We conclude this section with the definition of Ramanujan's general theta function [1, p. 34], $f(a,b)$ and its special cases.

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{\frac{k(k+1)}{2}} b^{\frac{k(k-1)}{2}}, \quad |ab| < 1.$$

Jacobi's triple product identity states that

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty.$$

In particular,

$$f(-q) = f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2}} = (q;q)_\infty, \tag{6}$$

$$\varphi(q) = f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2} \tag{7}$$

and

$$\psi(q) = f(q,q^3) = \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} = (-q;q)_\infty (q^2;q^2)_\infty = \frac{f_2^2}{f_1}. \tag{8}$$

Note that, for any positive integer k, $k(k+1)/2$ denotes the k^{th} triangular number.

2. Main results and Proofs

In this section, we establish several congruences for 5-core partitions.

Theorem 2.1. *For all integers $\alpha, \beta, n \geq 0$ and $r \in \{7, 11, 13, 14\}$,*

$$\sum_{n=0}^{\infty} a_5(3^{\alpha+1} \cdot 5^{\beta} n + 3^{\alpha} \cdot 5^{\beta} - 1) q^n \equiv (-1)^{\kappa(\alpha)} f_3 f_5 \pmod{3}, \tag{9}$$

$$\sum_{n=0}^{\infty} a_5(3^{\alpha+1} \cdot 5^{\beta} n + 2 \cdot 3^{\alpha} \cdot 5^{\beta} - 1) q^n \equiv (-1)^{\kappa(\alpha)} f_1 f_{15} \pmod{3} \tag{10}$$

and

$$a_5(3^{\alpha+1} \cdot 5^{\beta+1} n + r \cdot 3^{\alpha} \cdot 5^{\beta} - 1) \equiv 0 \pmod{3}, \tag{11}$$

where

$$\kappa(\alpha) = \begin{cases} \alpha & \text{if } \beta \text{ is even}, \\ \alpha + 1 & \text{if } \beta \text{ is odd}. \end{cases}$$

Proof. From the binomial theorem, we can see that for all positive integers k and m,

$$f_k^{3m} \equiv f_{3k}^{m} \pmod{3}. \tag{12}$$

Using (12),

$$\frac{f_5^5}{f_1} \equiv \frac{f_1^2 f_5^2 f_{15}}{f_3} \pmod{3}.$$

In view of (1) and the above identity,

$$\sum_{n=0}^{\infty} a_5(n) q^n \equiv \frac{f_1^2 f_5^2 f_{15}}{f_3} \pmod{3}. \tag{13}$$

Liu and Wang [9, Lemma 4] have shown that

$$f_1^2 f_5^2 \equiv f_3^4 + q f_3^2 f_{15}^2 - q^2 f_{15}^4 \pmod{3}. \tag{14}$$

Using (13) and (14), Das [3] obtained

$$\sum_{n=0}^{\infty} a_5(n) q^n \equiv f_9 f_{15} + q f_3 f_{45} - q^2 \frac{f_{15}^5}{f_3} \pmod{3}, \tag{15}$$

which implies that

$$\sum_{n=0}^{\infty} a_5(3n) q^n \equiv f_3 f_5 \pmod{3}, \tag{16}$$

$$\sum_{n=0}^{\infty} a_5(3n+1) q^n \equiv f_1 f_{15} \pmod{3} \tag{17}$$

and

$$\sum_{n=0}^{\infty} a_5(3n+2) q^n \equiv -\frac{f_5^5}{f_1} \pmod{3}. \tag{18}$$

From (1), (18) and mathematical induction, we have

$$a_5(3^{\alpha} n + 3^{\alpha} - 1) \equiv (-1)^{\alpha} a_5(n) \pmod{3}. \tag{19}$$

It follows from (16) and (19) that for all integers $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} a_5(3^{\alpha+1}n + 3^{\alpha} - 1)q^n \equiv (-1)^{\alpha} f_3 f_5 \pmod 3,$$

which is the $\beta = 0$ case of (9).

Ramanujan [12, p. 212] stated the following identity without proof:

$$f_1 = f_{25}\left(a(q) - q - q^2 a^{-1}(q)\right), \quad \text{where } a(q) := \frac{f(-q^{15}, -q^{10})}{f(-q^{20}, -q^5)}. \tag{20}$$

Suppose that the congruences (9) and (10) hold for some integer $\beta \geq 0$. Using (20),

$$\begin{aligned} &\sum_{n=0}^{\infty} a_5(3^{\alpha+1} \cdot 5^{\beta} n + 3^{\alpha} \cdot 5^{\beta} - 1)q^n \\ &\equiv (-1)^{\kappa(\alpha)} f_5 f_{75}\left(a(q^3) - q^3 - q^6 a^{-1}(q^3)\right) \pmod 3 \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} a_5(3^{\alpha+1} \cdot 5^{\beta} n + 2 \cdot 3^{\alpha} \cdot 5^{\beta} - 1)q^n \\ &\equiv (-1)^{\kappa(\alpha)} f_{15} f_{25}\left(a(q) - q - q^2 a^{-1}(q)\right) \pmod 3. \end{aligned} \tag{22}$$

Extracting the terms containing q^{5n+3} from (21) yields

$$\begin{aligned} &\sum_{n=0}^{\infty} a_5(3^{\alpha+1} \cdot 5^{\beta}(5n+3) + 3^{\alpha} \cdot 5^{\beta} - 1)q^n \\ &= \sum_{n=0}^{\infty} a_5(3^{\alpha+1} \cdot 5^{\beta+1} n + 2 \cdot 3^{\alpha} \cdot 5^{\beta+1} - 1)q^n \\ &\equiv (-1)^{\kappa(\alpha)+1} f_1 f_{15} \pmod 3. \end{aligned} \tag{23}$$

Now, extracting the terms containing q^{5n+1} from (22) yields

$$\begin{aligned} &\sum_{n=0}^{\infty} a_5(3^{\alpha+1} \cdot 5^{\beta}(5n+1) + 2 \cdot 3^{\alpha} \cdot 5^{\beta} - 1)q^n \\ &= \sum_{n=0}^{\infty} a_5(3^{\alpha+1} \cdot 5^{\beta+1} n + 3^{\alpha} \cdot 5^{\beta+1} - 1)q^n \\ &\equiv (-1)^{\kappa(\alpha)+1} f_3 f_5 \pmod 3. \end{aligned} \tag{24}$$

Thus, (21) and (22) are true for $\beta + 1$. Hence by induction (9) and (10) are true for any β. Thanks to (20), congruence (11) follows from (9) and (10). □

Theorem 2.2. *For all integers* $\alpha, \beta, n \geq 0, s \in \{11, 13, 17, 19\}$ *and* $t \in \{17, 33\}$, *we have*

(25)
$$\sum_{n=0}^{\infty} a_5(2^{\alpha+2} \cdot 5^{2\beta+1} n + 2^{\alpha} \cdot 5^{2\beta+1} - 1) q^n \equiv 2\psi(q^2) - (-1)^{\kappa(\alpha)} f_1 f_5 \pmod{4},$$

(26)
$$\sum_{n=0}^{\infty} a_5(2^{\alpha+2} \cdot 5^{\beta} n + 3 \cdot 2^{\alpha} \cdot 5^{\beta} - 1) q^n \equiv 2\psi(q)\psi(q^5) \pmod{4},$$

(27) $$a_5(2^{\alpha+2} \cdot 5^{\beta+1} n + s \cdot 2^{\alpha} \cdot 5^{\beta} - 1) \equiv 0 \pmod{4}$$

and

(28) $$a_5(5 \cdot 2^{\alpha+3} n + t \cdot 2^{\alpha} - 1) \equiv 0 \pmod{8},$$

where

$$\kappa(\alpha) = \begin{cases} 0 & \textit{if } \alpha \textit{ is } 0, \\ \alpha & \textit{otherwise.} \end{cases}$$

Proof. From the binomial theorem, we see that for all positive integers k and m,

(29) $$f_k^{2^m} \equiv f_{2k}^{2^{m-1}} \pmod{2^m}.$$

By (29),

(30) $$\frac{f_5^5}{f_1} \equiv \frac{f_{10}^4}{f_1 f_5^3} \pmod{8}.$$

From (1) and (29),

(31) $$\sum_{n=0}^{\infty} a_5(n) q^n \equiv \frac{f_{10}^4}{f_1 f_5^3} \pmod{8}.$$

Let

$$\sum_{n=0}^{\infty} p_{[1^1 5^3]}(n) q^n = \frac{1}{f_1 f_5^3}.$$

From [10, Corollary 4.2.1],

(32) $$\sum_{n=0}^{\infty} p_{[1^1 5^3]}(2n) q^n = 2q \frac{f_2^2 f_{10}^6}{f_1^3 f_5^9} + \frac{f_2 f_{10}^3}{f_5^8}.$$

Substituting (32) in (31) and using (29),

(33) $$\sum_{n=0}^{\infty} a_5(2n) q^n \equiv 2q \frac{f_2^2 f_{10}^6}{f_1^3 f_5^5} + \frac{f_2 f_{10}^3}{f_5^4} \equiv 2q \frac{f_1 f_{10}^4}{f_5} + \frac{f_2 f_{10}^3}{f_5^4} \pmod{8}.$$

Hirschhorn and Sellers [8] have proved that

(34) $$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}.$$

Replacing q by $-q$ in (34) and using the relation

$$(-q; -q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we obtain

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{35}$$

The following relation is an immediate consequence of dissection formulas of Ramanujan, collected in Berndt's book [1, Entry 25, p. 40].

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{36}$$

Substituting (35) and (36) in (33) and extracting the terms involving even and odd powers of q,

$$\sum_{n=0}^{\infty} a_5(4n)q^n = -2q\frac{f_2^2 f_5^2 f_{20}}{f_4} + \frac{f_1 f_{10}^{14}}{f_5^{11} f_{20}^4} \tag{37}$$

and

$$\sum_{n=0}^{\infty} a_5(4n+2)q^n = 2\frac{f_1 f_4 f_5 f_{10}^3}{f_2 f_{20}} + 4q^2\frac{f_1 f_{10}^2 f_{20}^4}{f_5^7}. \tag{38}$$

From [1, Corollary (i), p. 49],

$$\varphi(-q) = \varphi(-q^{25}) - 2qf(-q^{15}, -q^{35}) + 2q^4 f(-q^5, -q^{45}). \tag{39}$$

By (20), (39) and (37),

(40)

$$\sum_{n=0}^{\infty} a_5(4n)q^n \equiv -2qf_5^2 f_{20}\left(\varphi(-q^{50}) - 2q^2 f(-q^{30}, -q^{70}) + 2q^8 f(-q^{10}, -q^{90})\right) + \frac{f_{10}^2 f_{25}}{f_5^3}\left(a(q) - q - q^2 a^{-1}(q)\right) \pmod 8.$$

Thus,

$$\sum_{n=0}^{\infty} a_5(20n+12)q^n \equiv 4f_1^2 f_4 f(-q^6, -q^{14}) \pmod 8, \tag{41}$$

$$\sum_{n=0}^{\infty} a_5(20n+16)q^n \equiv -4qf_1^2 f_4 f(-q^2, -q^{18}) \pmod 8 \tag{42}$$

and

$$\sum_{n=0}^{\infty} a_5(20n+4)q^n \equiv -2f_1^2 f_4\, \varphi(-q^{10}) - \frac{f_2^2 f_5}{f_1^3} \pmod 8, \tag{43}$$

which yields,

$$a_5(20n+12) \equiv 0 \pmod 4, \tag{44}$$

$$a_5(20n+16) \equiv 0 \pmod 4 \tag{45}$$

for all $n \geq 0$ and

$$\sum_{n=0}^{\infty} a_5(20n+4)q^n \equiv 2\psi(q^2) - f_1 f_5 \pmod 4. \tag{46}$$

By setting $p = 2$ and replacing n by $2n$ in (2), we obtain

$$a_5(2^{\alpha+1}n + 2^{\alpha} - 1) = \frac{2^{\alpha+1} + (-1)^{\alpha}}{3} a_5(2n). \tag{47}$$

It follows from (46) and (47) that for all integers $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} a_5(5 \cdot 2^{\alpha+2}n + 5 \cdot 2^{\alpha} - 1)q^n \equiv 2\psi(q^2) - (-1)^{\kappa(\alpha)} f_1 f_5 \pmod{4}, \tag{48}$$

which is the $\beta = 0$ case of (25).

From [1, Corollary (ii), p. 49],

$$\psi(q) = f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25}). \tag{49}$$

Now suppose the congruence (25) is true for some $\beta \geq 0$. Utilizing (20) and (49) in (25) and extracting the terms involving q^{5n+1} yields,

$$\begin{aligned}\sum_{n=0}^{\infty} & a_5(2^{\alpha+2} \cdot 5^{2\beta+2}n + 2^{\alpha} \cdot 5^{2\beta+2} - 1)q^n \\ & \equiv 2q\psi(q^{10}) + (-1)^{\kappa(\alpha)} f_1 f_5 \pmod{4}.\end{aligned} \tag{50}$$

Substituting (20) in (50) and extracting the terms containing q^{5n+1},

$$\begin{aligned}\sum_{n=0}^{\infty} & a_5(2^{\alpha+2} \cdot 5^{2\beta+3}n + 2^{\alpha} \cdot 5^{2\beta+3} - 1)q^n \\ & \equiv 2\psi(q^2) - (-1)^{\kappa(\alpha)} f_1 f_5 \pmod{4},\end{aligned}$$

which implies that (25) is true for $\beta + 1$.

Using (29), we can rewrite (38) as

$$\sum_{n=0}^{\infty} a_5(4n+2)q^n \equiv 2\psi(q)\psi(q^5) \pmod{4}. \tag{51}$$

By (47) and (51),

$$\sum_{n=0}^{\infty} a_5(2^{\alpha+2}n + 3 \cdot 2^{\alpha} - 1)q^n \equiv 2\psi(q)\psi(q^5) \pmod{4}. \tag{52}$$

It is clear that (26) follows from (52), (49) and mathematical induction. Thanks to (20) and (49), congruence (27) follows from (44), (45), (25), (50) and (26).

Using (29) in (41) and (42), we can easily see that for all $n \geq 0$,

$$a_5(40n + 32) \equiv 0 \pmod{8} \tag{53}$$

and

$$a_5(40n + 16) \equiv 0 \pmod{8}. \tag{54}$$

Congruence (28) follows from (53), (54) and (47). □

Proof of the Theorem 1.2. Replacing n by $2^{4\alpha+2+i}n + \dfrac{u \cdot 2^{4\alpha+i} - r}{15}$ in (11) such that $u \cdot 2^i \equiv r \pmod{15}$, where $i \in \{0, 1, 2, 3\}$, we have

$$a_5(2^{\alpha+2} \cdot 3^{\beta+1} \cdot 5^{\gamma+1}n + u \cdot 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma} - 1) \equiv 0 \pmod{3}. \tag{55}$$

Again, replacing n by $3^{4\beta+1+i}n + \frac{u\cdot 3^{4\beta+i}-s}{20}$ in (27) such that $u\cdot 3^i \equiv s \pmod{20}$, where $i \in \{0,1,2,3\}$, we have

$$a_5(2^{\alpha+2}\cdot 3^{\beta+1}\cdot 5^{\gamma+1}n + u\cdot 2^{\alpha}\cdot 3^{\beta}\cdot 5^{\gamma} - 1) \equiv 0 \pmod{4}. \tag{56}$$

Thus, congruence (4) immediately follows from (55) and (56). Similarly, congruence (5) follows from (11) and (28). □

Theorem 2.3. *If n cannot be represented as a sum of a triangular number and five times a triangular number, then*

$$a_5(4n+2) \equiv 0 \pmod{4}.$$

Proof. Using (8) in (51), we have

$$\sum_{n=0}^{\infty} a_5(4n+2)q^n \equiv 2\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q^{\frac{k(k+1)}{2}+5\frac{m(m+1)}{2}} \pmod{4}. \tag{57}$$

Theorem 2.3 follows from (57). □

Corollary 2.4. *For any positive integer k, let $p_j > 5,\ 1 \le j \le k$ be primes. If $\left(\frac{-5}{p_j}\right) = -1$ for every j, then for all nonnegative integers n with $p_k \nmid n$,*

$$a_5(4p_1^2p_2^2\cdots p_{k-1}^2p_kn + 3p_1^2p_2^2\cdots p_k^2 - 1) \equiv 0 \pmod{4}. \tag{58}$$

Proof. By (57),

$$\sum_{n=0}^{\infty} a_5(4n+2)q^{8n+6} \equiv 2\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q^{(2k+1)^2+5(2m+1)^2} \pmod{4}, \tag{59}$$

which implies that if $8n+6$ is not of the form x^2+5y^2, then $a_5(4n+2) \equiv 0 \pmod{4}$. Let $\vartheta_{p_j}(N)$ denotes the highest power of p_j dividing N. If N is of the form x^2+5y^2, then $\vartheta_{p_j}(N)$ is always even, since $(\frac{-5}{p_j}) = -1$. Let

$$\begin{aligned} N &= 8\left(p_1^2p_2^2\cdots p_{k-1}^2p_kn + 3\frac{p_1^2p_2^2\cdots p_k^2-1}{4}\right)+6 \\ &= 8p_1^2p_2^2\cdots p_{k-1}^2p_kn + 6p_1^2p_2^2\cdots p_k^2. \end{aligned}$$

Since $p_k \nmid n$, $\vartheta_{p_k}(N) = 1$, which is an odd number and hence N is not of the form x^2+5y^2. Therefore (58) holds. □

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